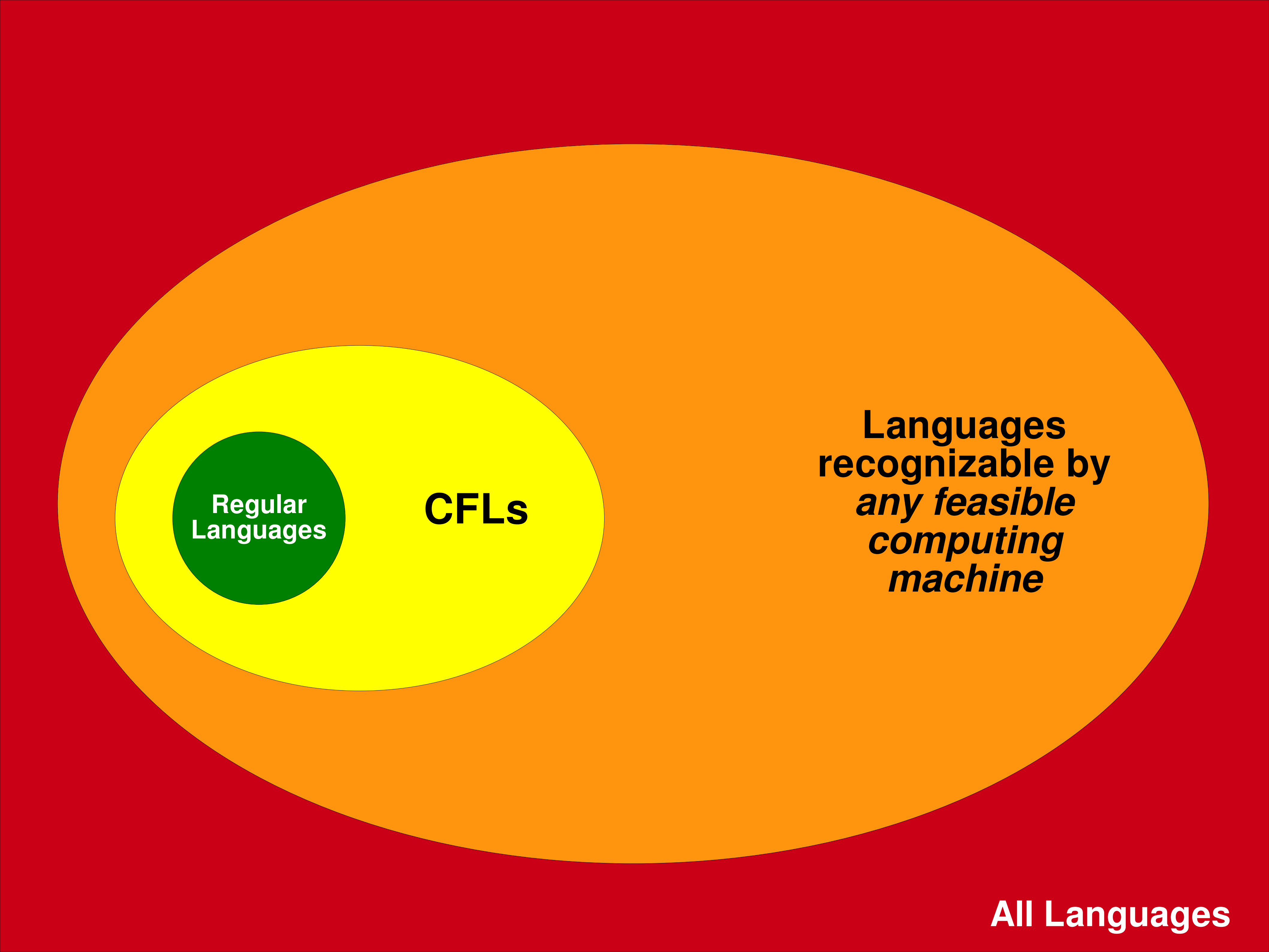
CS103
Winter 2025

Lecture 20:

Turing Machines

Part 1 of 3

What problems can we solve with a computer?


Regular Languages
CFLs
Languages recognizable by *any feasible computing machine*
All Languages

That same drawing, to scale.

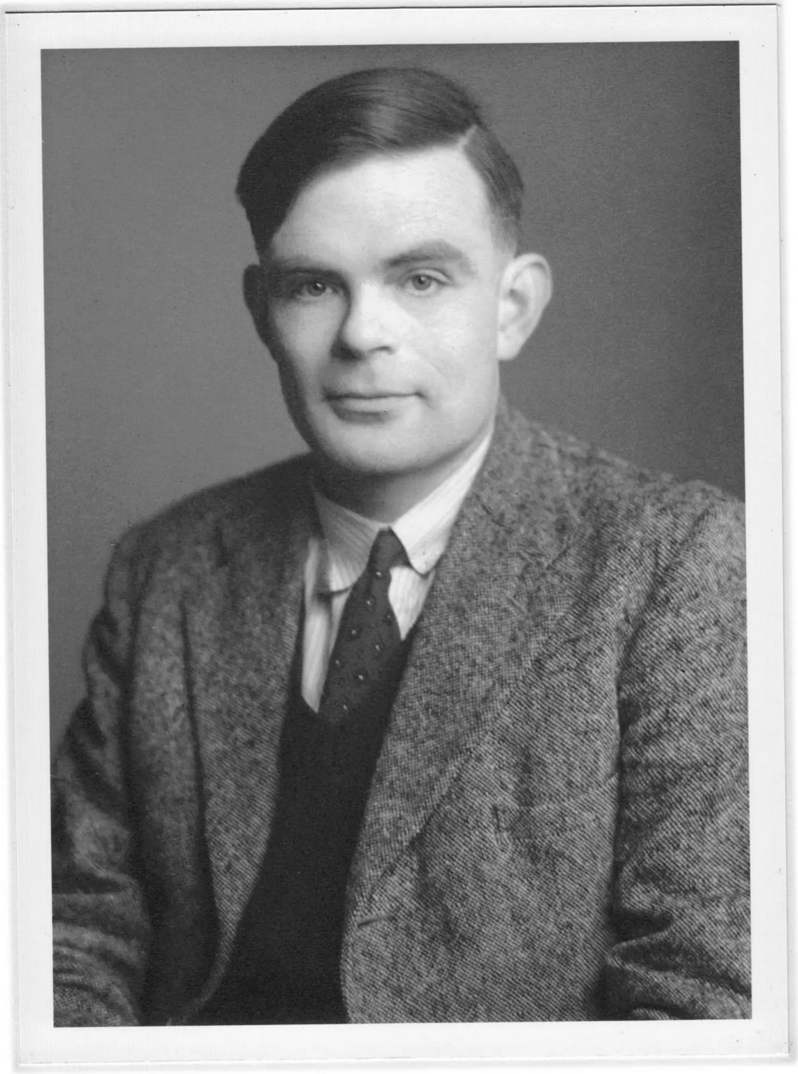
The Problem

- Finite automata accept precisely the regular languages.
- We may need unbounded memory to recognize context-free languages.
  - e.g. $\{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N} \}$ requires unbounded counting.
- How do we model a computing device that has unbounded memory?

A Brief History Lesson

Turing Machines

- In March 1936, Alan Turing (aged 23!) published a paper detailing the ***a-machine*** (for ***automatic machine***), an automaton for computing on real numbers.
- They're now more popularly referred to as ***Turing machines*** in his honor.
- He also later made contributions to computational biology, artificial intelligence, cryptography, etc. Seriously, Google this guy.



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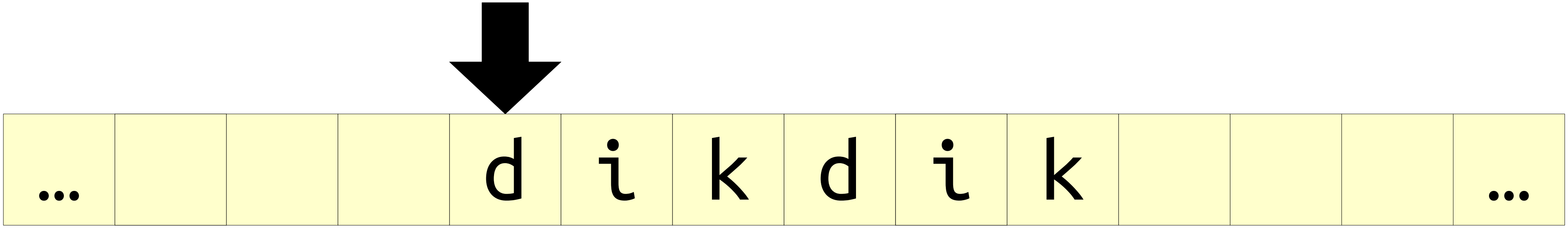
$$
\begin{array}{cccccccccccccccc}
 & & & & & {\scriptstyle 1} & & {\scriptstyle 1} & & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & \\
 & 2 & 7 & 1 & 8 & 2 & 8 & 1 & 8 & 2 & 8 & 4 & 5 & 9 & 0 \\
+ & 3 & 1 & 4 & 1 & 5 & 9 & 2 & 6 & 5 & 3 & 5 & 8 & 9 & 7 \\
\hline
 & 5 & 8 & 5 & 9 & 8 & 7 & 4 & 4 & 8 & 2 & 0 & 4 & 8 & 7
\end{array}
$$

Key Idea: Even if you need huge amounts of scratch space to perform a calculation, at each point in the calculation you only need access to a small amount of that scratch space.

Turing Machines

- To provide his machines extra memory, Turing gave his machines access to an ***infinite tape*** subdivided into a number of ***tape cells***.
- A Turing machine can only see one tape cell at a time, the one pointed at by the ***tape head***.
- The Turing machine can
  - read the cell under the tape head,
  - (possibly) change which symbol was written under the tape head, and
  - move its tape head to the left or to the right.



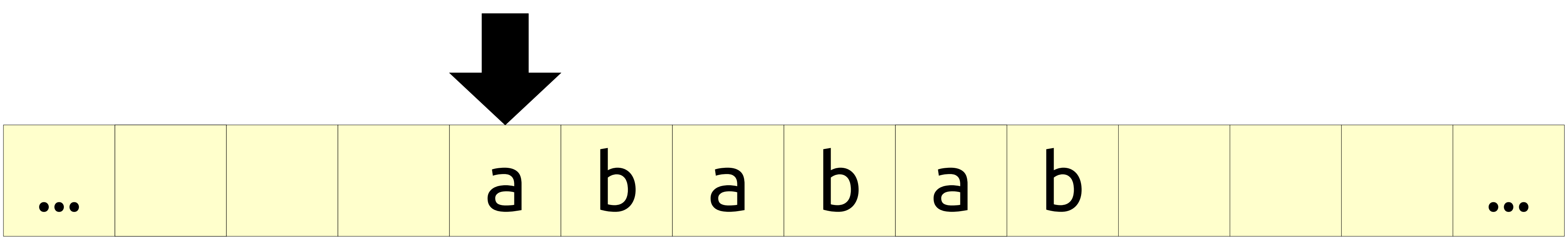
Turing Machines

- Over the years, there have been many simplifications and edits to Turing's original automata.
  - In practice, electronic computers are written in terms of individual instructions rather than states and transitions.
  - Turing's original paper deals with computing individual real numbers; we typically want to compute functions of inputs.
- What we're going to present as "Turing machines" in this class differ significantly from Turing's original description, while retaining the core essential ideas.
  - (Our model is closer to Emil Post's *Formulation 1* and Hao Wang's *Basic Machine B*, for those of you who are curious.)
- If you'd like to learn more about Turing's original version of the Turing machine, come chat with me after class!

Turing Machines

- A TM is a series of instructions that control a tape head as it moves across an infinite tape.
- The tape begins with the input string written somewhere, surrounded by infinitely many blank cells.
  - Rule: The input string cannot contain blank cells.
- The tape head begins above the first character of the input. (If the input is ε, the tape head points somewhere on a blank tape.)

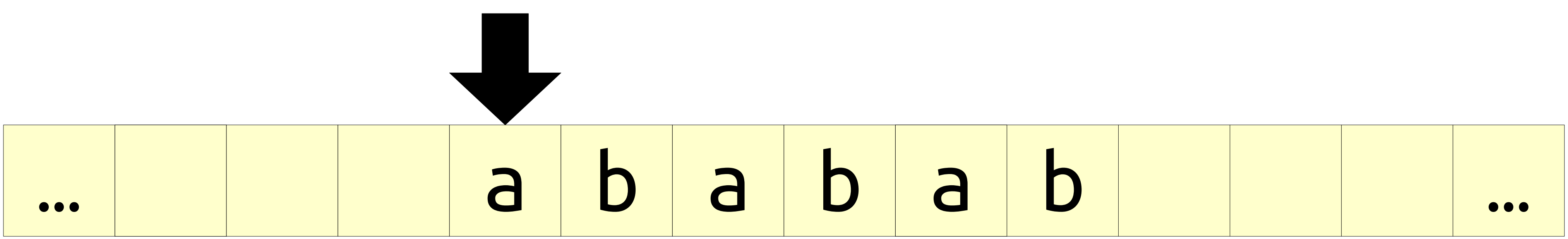
```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



Turing Machines

- We begin at the `Start` label.
- Labels indicate different sections of code. The name `Start` is special and means “begin here.”
- Labels have no effect when executed. We just move to the next line.

```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



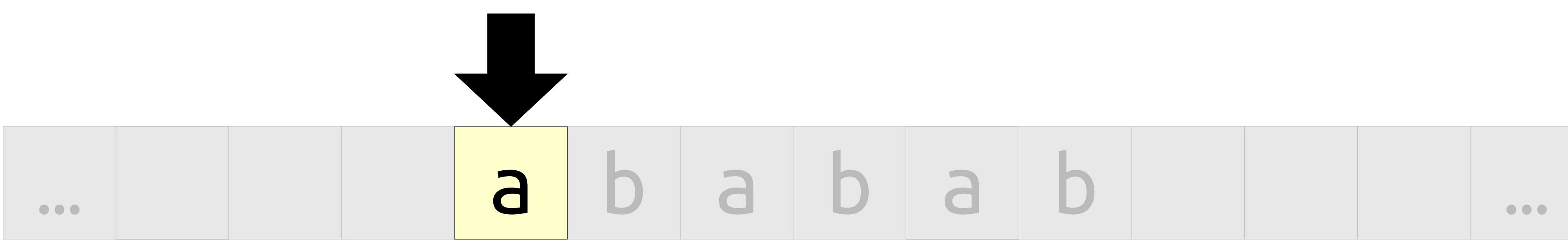
Turing Machines

- A statement of the form

  If *symbol command*

  checks if the character under the tape head is *symbol*.
- If so, it executes *command*.
- If not, nothing happens.

```
Start:
    If Blank Return True
    If 'b' Return False
    Write 'x'
    Move Right
    If Not 'b' Return False
    Write 'x'
    Move Right
    Goto Start
```



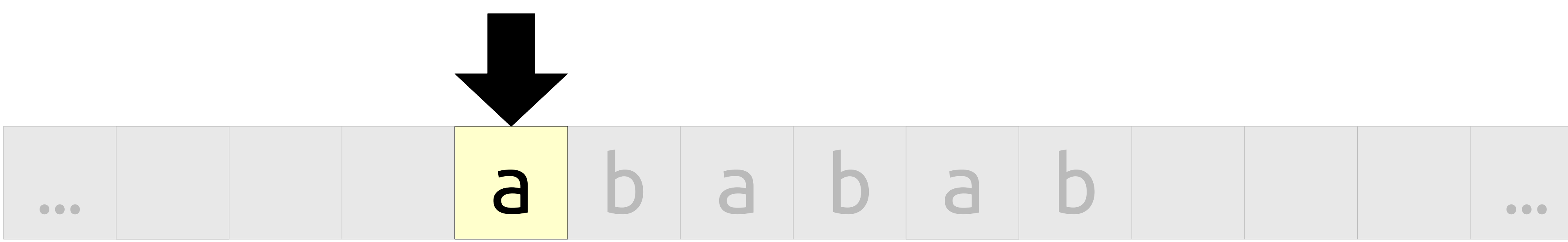
Turing Machines

- A statement of the form

  `If` *symbol command*

  checks if the character under the tape head is *symbol*.
- If so, it executes *command*.
- If not, nothing happens.

```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



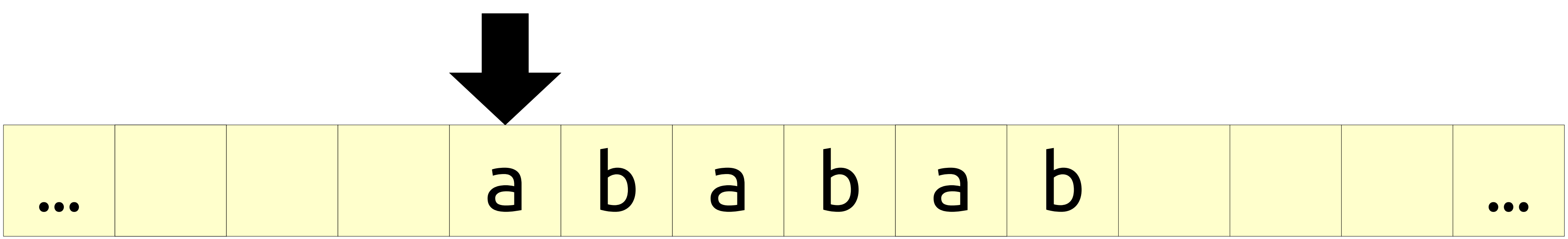
Turing Machines

- The statement

  `Write` *symbol*

  writes *symbol* to the cell under the tape head.
- The *symbol* can either be `Blank` or a character in quotes.

```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



Turing Machines

- The statement

  `Write` *symbol*

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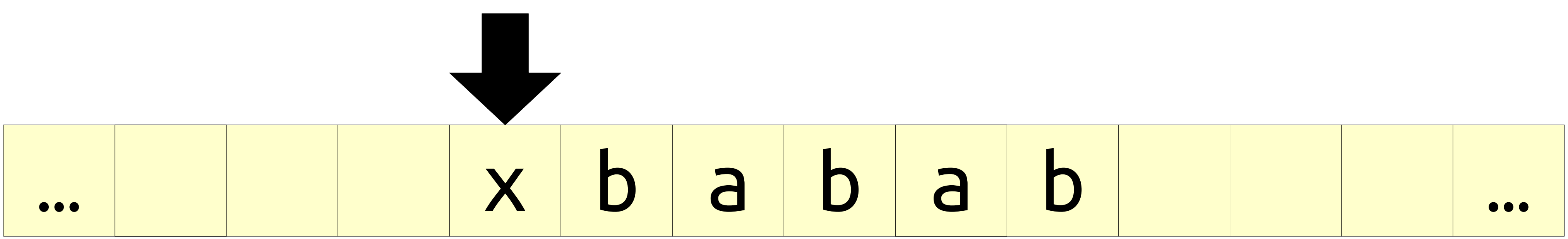
```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



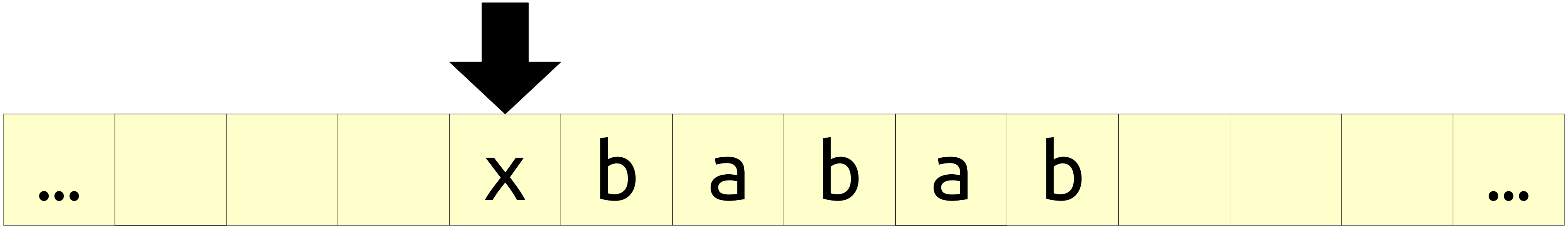
Turing Machines

- The command

  `Move` *direction*

  moves the tape head one step in the indicated direction (either `Left` or `Right`).

```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



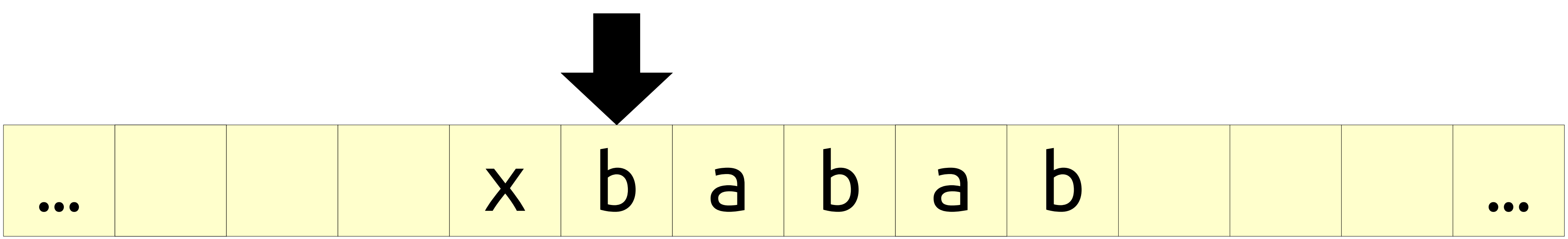
Turing Machines

- The command

  Move *direction*

  moves the tape head one step in the indicated direction (either Left or Right).

```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



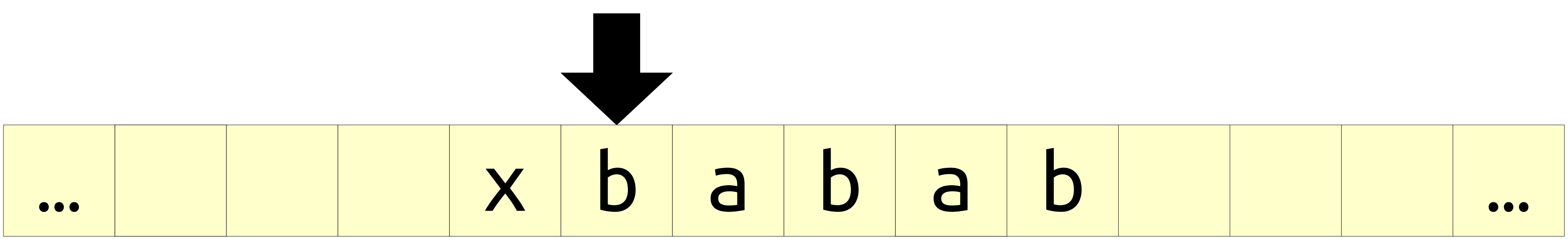
Turing Machines

- A statement of the form

  `If Not` *symbol command*

  sees if the cell under the tape head holds *symbol*.
- If so, nothing happens.
- If not, it executes *command*.

```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



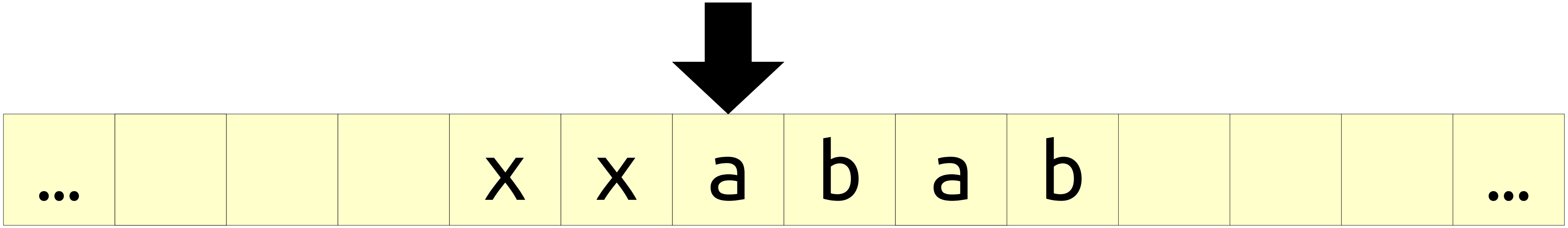
Turing Machines

- The command

  Goto *label*

  jumps to the indicated label.
- This program just has a `Start` label, but most interesting programs have other labels beyond this.

```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



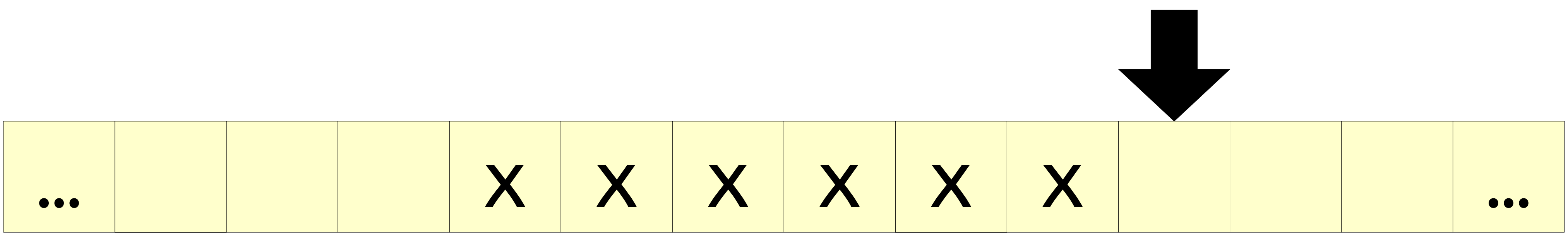
Turing Machines

- A TM stops when executing the

  `Return` *result*

  command.
- Here, *result* can be either `True` or `False`.
- (If we "fall off" the bottom of the program, the TM acts as though it executes the `Return False` command.)

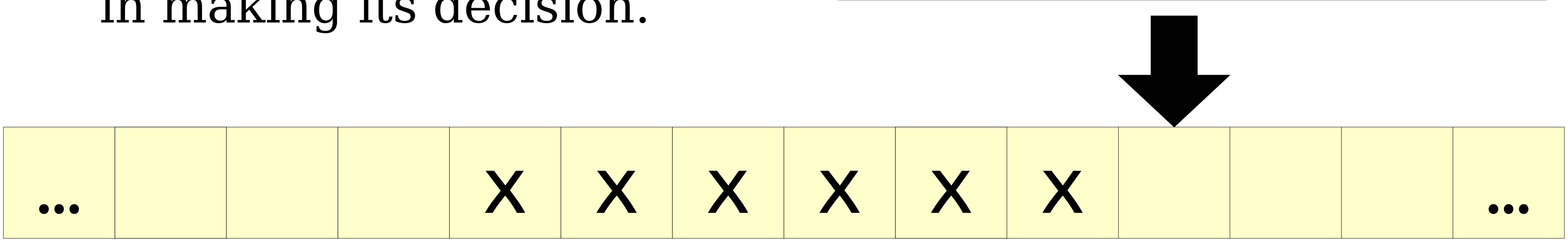
```
Start:
    If Blank Return True
    If 'b' Return False
    Write 'x'
    Move Right
    If Not 'b' Return False
    Write 'x'
    Move Right
    Goto Start
```



Turing Machines

- This TM initially started up with the string ababab on its tape, so this means that TM returns true on the input ababab, not xxxxxx.
- An intuition for this: we gave this program an input. It therefore returned true with respect to that input, not whatever internal data it generated in making its decision.

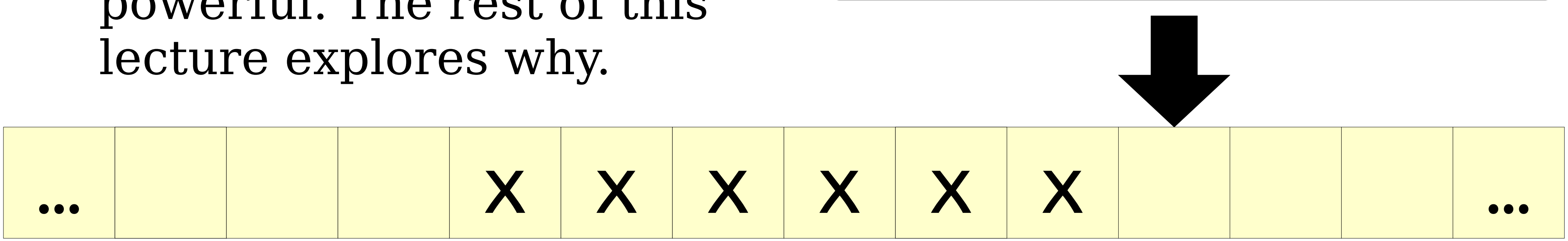
```
Start:
    If Blank Return True
    If 'b' Return False
    Write 'x'
    Move Right
    If Not 'b' Return False
    Write 'x'
    Move Right
    Goto Start
```



Turing Machines

- To summarize, we only have six commands:
  - `Move` *direction*
  - `Write` *symbol*
  - `Goto` *label*
  - `Return` *result*
  - `If` *symbol command*
  - `If Not` *symbol command*
- Despite their simplicitly, TMs are *surprisingly* powerful. The rest of this lecture explores why.

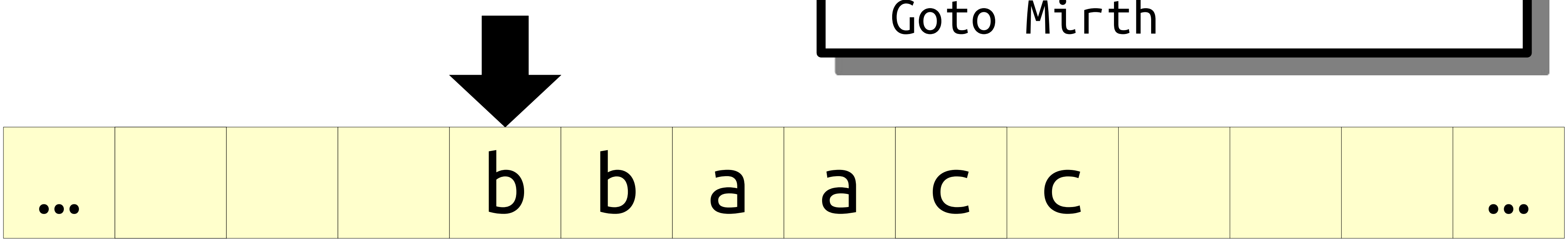
```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```



Your Turn!

- Draw what the tape and tape head look like when this TM finishes running.
- Is the input `bbaacc` accepted or rejected?
- More generally, what does this TM do?

```
Start:
   If 'a' Goto Mirth
   If Blank Return False
   Move Right
   Goto Start
Mirth:
   If 'b' Return True
   If Blank Return False
   Move Right
   Goto Mirth
```



Programming Turing Machines

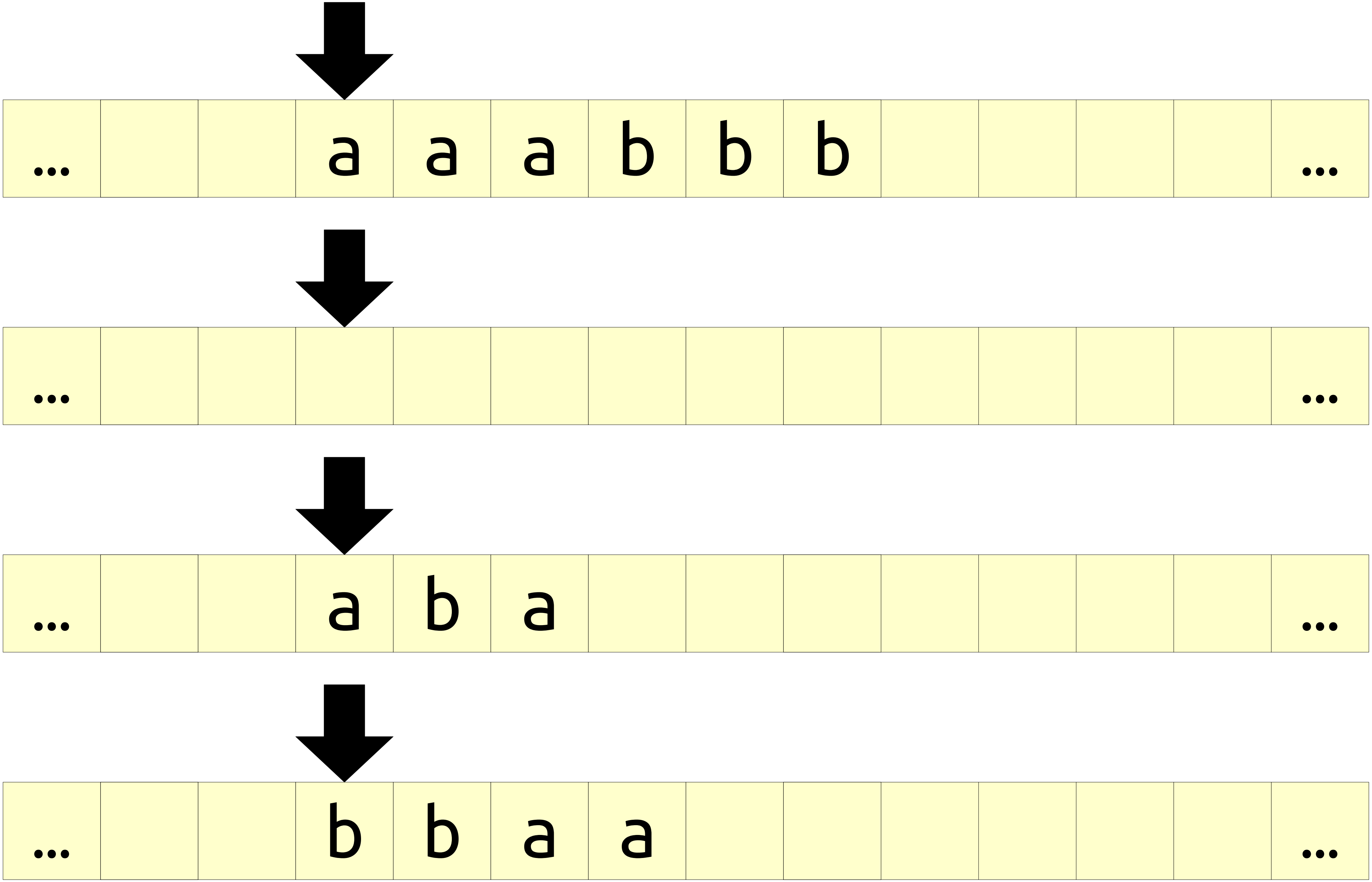
Our First Challenge

- The language

$$\{ \mathtt{a}^n \mathtt{b}^n \mid n \in \mathbb{N} \}$$

  is a canonical example of a nonregular language. It's not possible to check if a string is in this language given only finite memory.
- Turing machines, however, are powerful enough to do this. Let's see how.

$$L = \{ \mathbf{a}^n \mathbf{b}^n \mid n \in \mathbb{N} \}$$



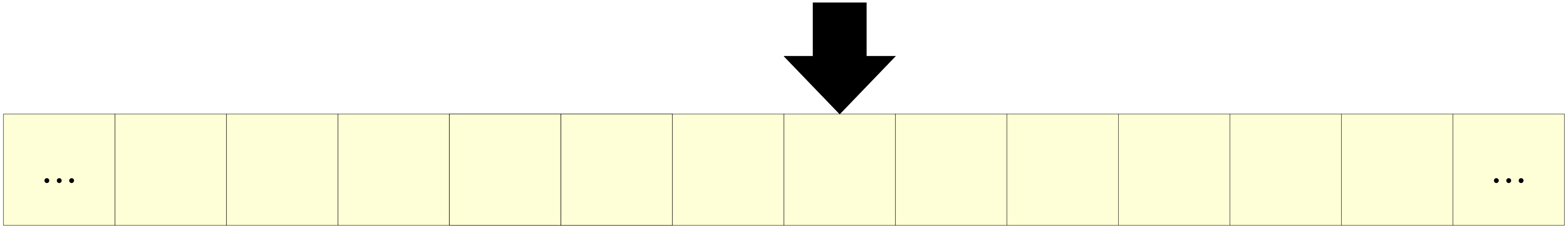
A Recursive Approach

- We can process our string using this recursive approach:
  - The string ε is in L.
  - The string **a**w**b** is in L if and only if w is in L.
  - Any string starting with **b** is not in L.
  - Any string ending with **a** is not in L.
- All that's left to do now is write a TM that implements this.

```
Start:
  If Blank Return True
  If 'b' Return False
  Write Blank

ZipRight:
  Move Right
  If Not Blank Goto ZipRight
  Move Left
  If Not 'b' Return False
  Write Blank

ZipLeft:
  Move Left
  If Not Blank Goto ZipLeft
  Move Right
  Goto Start
```



Time-Out for Announcements!

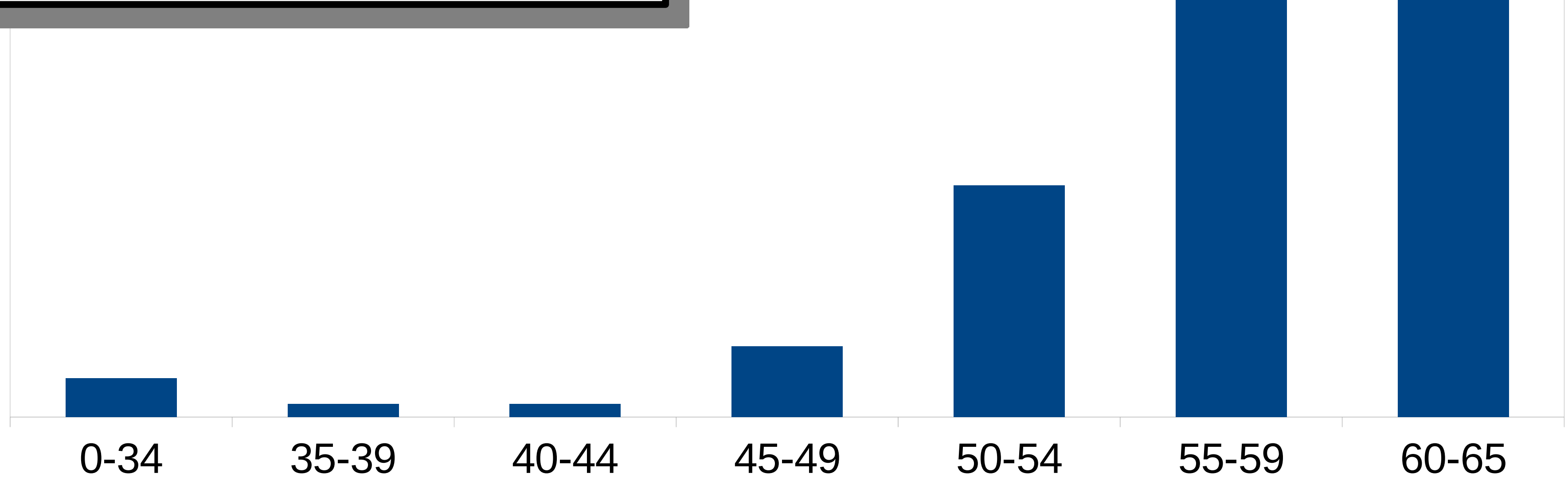
The State of Things

- Exam grading this weekend.
- Exam solutions and grades to be posted early next week.
- ***Do not withdraw or change your grading basis*** unless you have run some projections about your raw score!

Problem Set 6 Graded

- Regrade requests run Friday through Wednesday.

75th Percentile: **61/ 65 (94%)**
50th Percentile: **58 / 65 (89%)**
25th Percentile: **54 / 65 (83%)**



Back to CS103!

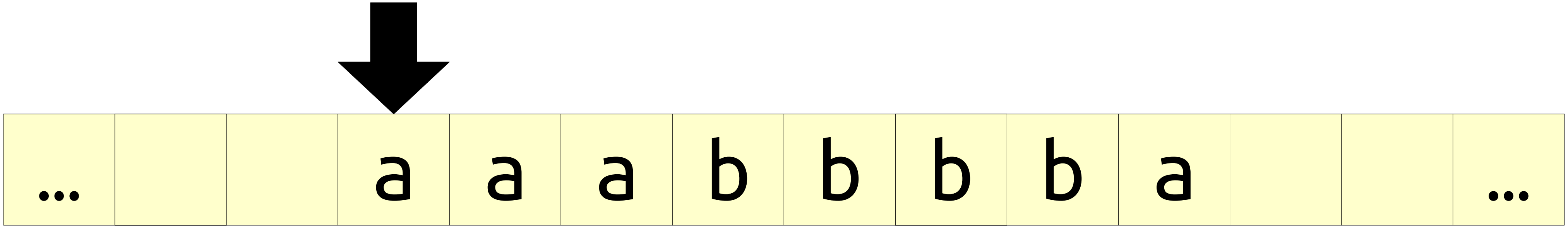
Our Next Challenge

- Let's now take aim at this more general language:

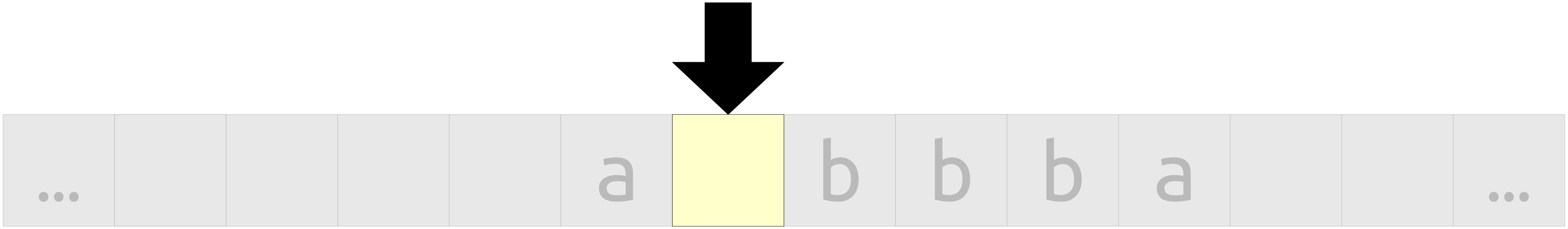
  $\{ w \in \{$`a`, `b`$\}^* \mid w$ has an equal number of `a`'s and `b`'s $\}$

- This language is not regular (do you see why?)
- It is context-free, but it's a bit tricky to write a CFG for it. (See PS8!)
- Let's see how to design a TM for it.

A Caveat

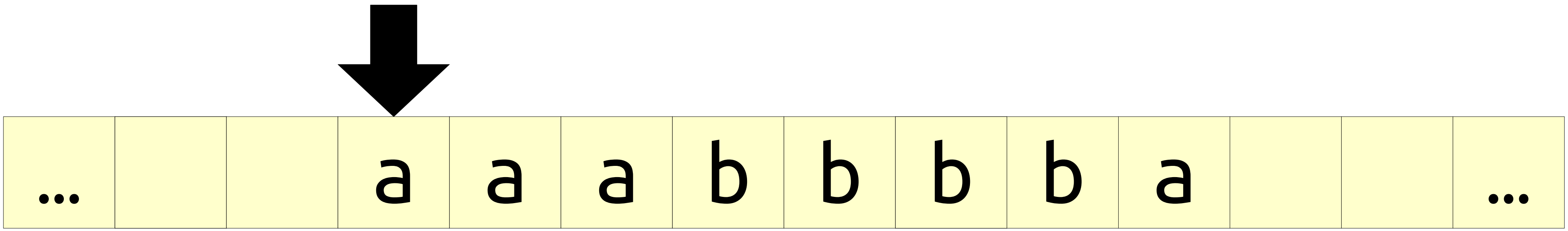


A Caveat

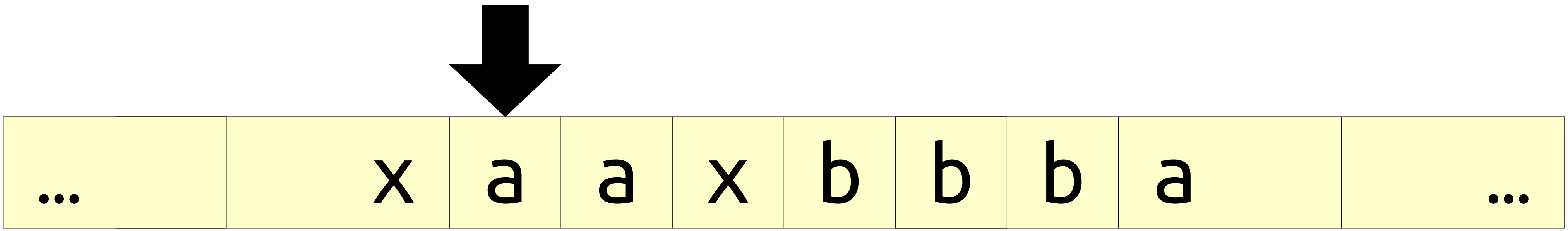


How do we know that this blank isn't one of the infinitely many blanks after our input string?

One Solution



One Solution

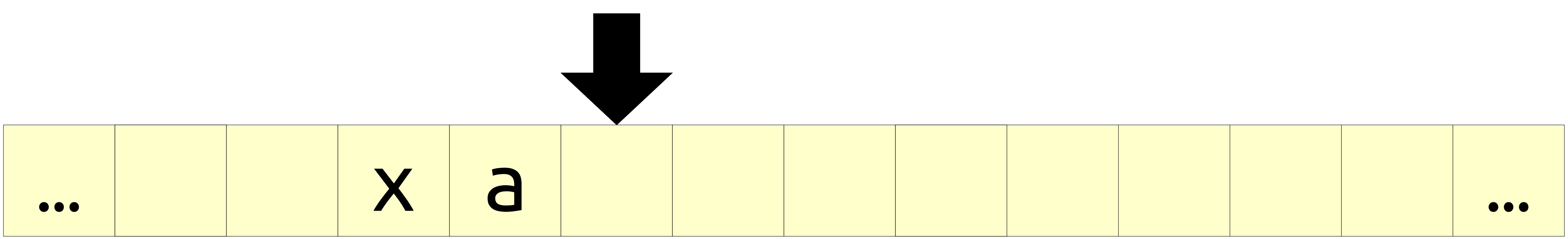


```
Start:
  If 'a' Goto FoundA
  If 'b' Goto FoundB
  If Blank Return True
  Move Right
  Goto Start

FoundA:
  Write 'x'
LoopA:
  Move Right
  If 'a' Goto LoopA
  If 'x' Goto LoopA
  If Blank Return False
  Write 'x'
  Goto GoHome

GoHome:
  Move Left
  If Not Blank Goto GoHome
  Move Right
  Goto Start

FoundB:
  Write 'x'
LoopB:
  Move Right
  If 'b' Goto LoopB
  If 'x' Goto LoopB
  If Blank Return False
  Write 'x'
  Goto GoHome
```



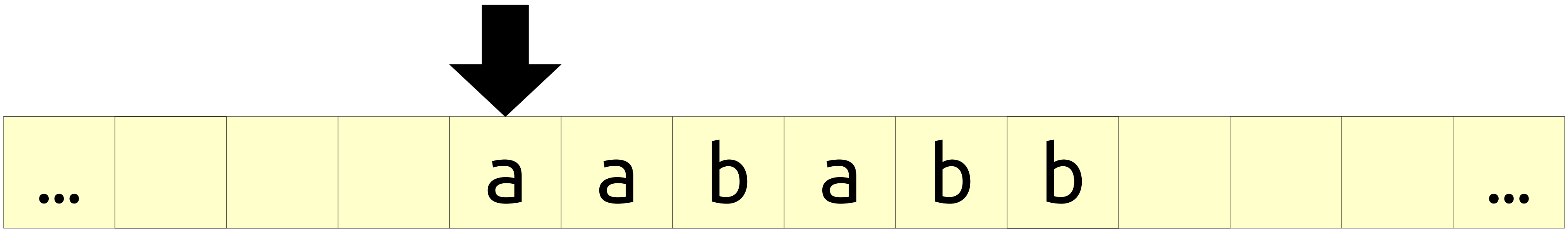
Another Idea

- We just built a TM for the language

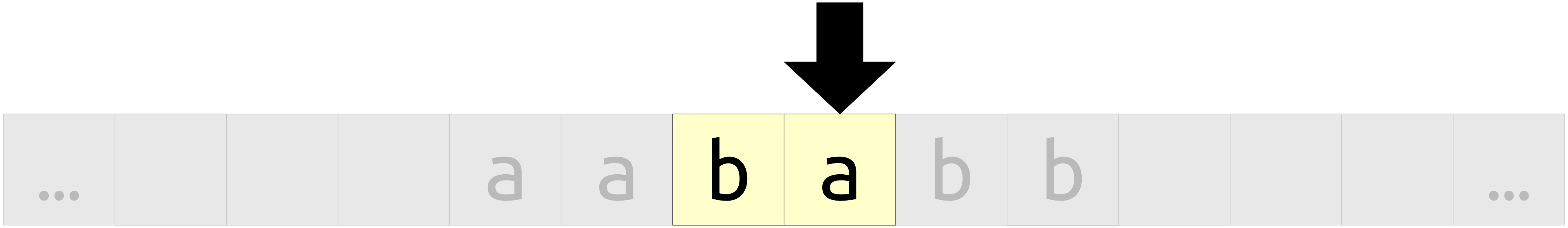
  { w ∈ {**a**, **b**}* | w has the same number of **a**'s and **b**'s }.

- An observation: this would be a *lot* easier to test for if all the **a**'s came before all the **b**'s.
  - In fact, that would turn this into checking if the string has the form $\mathtt{a}^n\mathtt{b}^n$, which we already know how to do!
- ***Idea:*** Could we sort the characters of our input string?

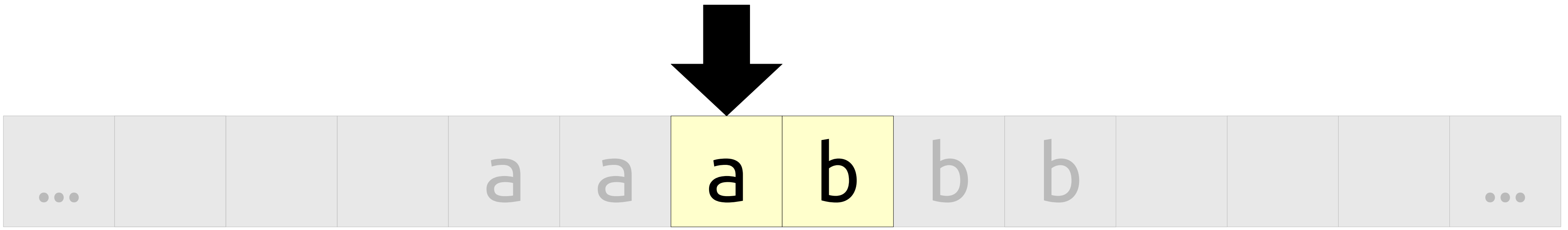
The Idea



The Idea



The Idea



Exploring This Idea

Cool TM Tricks 2: ***Decimal Fibonacci***

Summary for Today

- Turing machines are abstract computers that issue commands to an infinite tape subdivided into cells.
- Each step of the TM can move the tape head, change what’s on the tape, or jump to a different part of the program.
- TMs can be composed together to build larger TMs out of smaller ones.

Next Time

- ***The Church-Turing Thesis***
  - How powerful are Turing machines?
- ***Decidability and Recognizability***
  - Two notions of "solving a problem."